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Quark confinement and curved spaces

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Abstract. In this work it will be shown how quark confinement appears when wave equations derived in curved spaces are considered. First, the equations and their solutions for Coulomb-like potentials will be presented, and then, how this theory leads to quark confinement. A comparison between different models of confinement will also be made.

1 Introduction

The introduction of quarks in the physical theory in 1964, by the Gell-Mann [1] and Zweig [2] hypothesis, has also introduced the annoying question of quark confinement. If on the one hand this theory was able to organize and to explain the main proprieties of the hadrons with the scheme called the eightfold way [3], on the other hand, the impossibility of observing free quarks could be considered as a major problem. Since then, many authors proposed models in order to describe the structure of the hadrons in terms of confined quarks. A successful way to implement these ideas is to consider non-relativistic constituent quark models, such as the non-relativistic oscillator model, proposed by Dalitz in 1967 [4] and by Faiman in 1968 [5], where the baryons are supposed to be systems composed of three constituent quarks, confined by an harmonic oscillator potential, with the states determined by the Hamiltonian

$$H_0 = \sum_i \frac{p_i^2}{2m_i} + \frac{K}{2} \sum_{i>j} (\boldsymbol{r}_i - \boldsymbol{r}_j)^2 . \tag{1}$$

This model was improved by De Rújula [6] with the addition of a quark–quark spin-dependent potential, by Karl and Isgur, with the introduction of a quantum chromodynamics inspired potential [7] and also by Murthy [8], who considered a deformed oscillator. Heavy $q\bar{q}$ systems are equally well described by non-relativistic constituent quark models, such as the Cornell model [9, 10], that uses a linear plus Coulomb potential of the type

$$V(r) = \frac{a}{r} + br, \tag{2}$$

where a and b are constants determined phenomenologically. Some models, as for example [11, 12], are based on other mechanisms, and generate different potentials.

Despite the success of the non-relativistic models in describing the hadrons proprieties, theoretically, it is more reasonable to think of quarks as relativistic particles. In 1968, Bogolioubov [13] considered the baryons as spherical cavities, and inside of them the three constituent quarks are Dirac particles, that generate a self-consistent mean field that was represented by a scalar potential

$$V(r) = 0 \quad \text{for} \quad r \le R,$$
$$V(r) = V_0 \quad \text{for} \quad r > R, \tag{3}$$

and quark confinement is achieved for $V_0 \to \infty$. Further development of these ideas led to the MIT bag model [14] where the vacuum pressure was included. Other models based on the Dirac equation [15–17], with r^n confining potentials, as in [17]

$$V(r) = V_0 + \lambda r, \tag{4}$$

may be found in the literature, and they also show good agreement with the experimental data.

In a recent paper [18], a relativistic wave equation based on the general relativity formulation has been derived. This theory has been constructed taking into account the effect of different kind of interactions (electromagnetic and strong) in the metric of the space-time. In this work, the main objective is to investigate the quark confinement with this theory. As it will be seen in the next sections, very interesting results can be obtained this way, and in many aspects these results are qualitatively different from the ones obtained in the previously cited models.

This paper has the following structure: In Sect. 2 a brief review of the theory and the solution of the equation for a Coulomb-like potential are shown, in Sect. 3, the quark confinement effect that comes from this theory is presented, and in Sect. 4, a comparison between the different confinement mechanisms discussed in this paper is made and the conclusions are drawn.

2 Quantum mechanics in curved space-time

The Einstein general theory of relativity is one great achievement in the understanding of Nature, and when

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applied to very large systems, such as planets or galaxies, gives very precise results. Taking this fact into account, fundamental questions may be asked, as for example why the general covariance principle does not apply to very small systems, such as atoms or elementary particles, and if the laws of physics depend on the size of the object. In quantum systems, the electromagnetic and strong interactions dominate and the gravitational interaction is negligible, as the masses of the considered particles are very small. So, the gravitational potential may be turned off, and then, the curvature of the space-time will be predominantly due to the other interactions (electromagnetic and strong).

With these aspects in mind, in [18] a theory was proposed, and an equation similar to the Dirac equation was derived. In this section, a brief revision of this theory, will be made, and some results, necessary to the development of this paper will be shown.

The simplest way to formulate this theory is to consider systems where spherical symmetry exists. In this case, the space-time is described by the Schwarzschild metric [19,20],

$$\mathrm{d}s^2 = \xi \mathrm{d}\tau^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \xi^{-1}\mathrm{d}r^2, \tag{5}$$

where the factor $\xi(r) = (1 + V(r)/m_0c^2)^2$ is determined by the interaction potential $V(r)$ and is a function only of r.

From the definition of the energy-momentum relations and the respective quantum operators (mathematical details may be found in [18]) in the given metric, the general relativistic equation for spin-1/2 particles

$$\frac{\mathrm{i}\hbar}{\xi}\frac{\partial}{\partial t}\Psi = \left(-\mathrm{i}\hbar c\, \boldsymbol{\alpha}.\boldsymbol{\nabla} + \beta m_0 c^2\right)\Psi \tag{6}$$

has been deduced [18], where Ψ is a four-component spinor. One must note that despite the fact that this theory is conceptually more complicated than the Dirac one, the final equation is very similar to the Dirac equation, which is surprisingly in accord with his simplicity ideal.

The spatial part of Ψ may be written as

$$\psi = \begin{pmatrix} F(r)\chi_k^\mu \\ \mathrm{i}G(r)\chi_{-k}^\mu \end{pmatrix}, \tag{7}$$

where χ_k^μ are the usual two-component spinors, and k is related with the angular momentum by

$$\begin{aligned} k &= l \quad \text{for} \quad j = l - 1/2, \\ k &= -l - 1 \quad \text{for} \quad j = l + 1/2. \end{aligned} \tag{8}$$

The radial part of (6) may be rewritten as a pair of coupled equations for the and the F and G functions

$$\begin{aligned} \sqrt{\xi}\frac{\mathrm{d}F}{\mathrm{d}r} + (1+k)\frac{F}{r} &= \left(\frac{E}{\sqrt{\xi}} + m_0\right)G, \\ \sqrt{\xi}\frac{\mathrm{d}G}{\mathrm{d}r} + (1-k)\frac{G}{r} &= -\left(\frac{E}{\sqrt{\xi}} - m_0\right)F. \end{aligned} \tag{9}$$

Considering a coulomb-like potential $V(r) = -\alpha Z/r$ the ξ function becomes

$$\xi = \left(1 - \frac{\alpha Z}{m_0 c^2\, r}\right)^2, \tag{10}$$

and inserting (10) in (9) and making the substitution $\rho = \beta r$, the equations may be put in the form

$$\begin{aligned} \xi\frac{\mathrm{d}F}{\mathrm{d}\rho} + \sqrt{\xi}(1+k)\frac{F}{\rho} &= \left(\frac{E}{\beta} + \sqrt{\xi}\frac{m_0}{\beta}\right)G, \\ \xi\frac{\mathrm{d}G}{\mathrm{d}\rho} + \sqrt{\xi}(1-k)\frac{G}{\rho} &= -\left(\frac{E}{\beta} - \sqrt{\xi}\frac{m_0}{\beta}\right)F. \end{aligned} \tag{11}$$

The equations may be solved by the Frobenius method, expressing the F and G functions as power series of the form

$$\begin{aligned} F &= \rho^s \sum_{n=0}^{N} a_n \rho^n \mathrm{e}^{-\rho}, \\ G &= \rho^s \sum_{n=0}^{N} b_n \rho^n \mathrm{e}^{-\rho}. \end{aligned} \tag{12}$$

Substituting this expressions in the equations we find that $s = 0$ and the relations between the coefficients are obtained:

$$\begin{aligned} a_1 &= \left[\frac{1+k+\alpha\beta}{\alpha\beta}\right]a_0 \\ b_1 &= \left[\frac{1-k+\alpha\beta}{\alpha\beta}\right]b_0, \end{aligned} \tag{13}$$

$$\begin{aligned} 2\alpha^2\beta^2 a_2 \quad & -\alpha\beta\,(3+k+\alpha\beta)\,a_1 \\ & + (1+k+2\alpha\beta)\,a_0 + \alpha m_0 b_0 = 0, \\ 2\alpha^2\beta^2 b_2 \quad & -\alpha\beta\,(3-k+\alpha\beta)\,a_1 \\ & + (1-k+2\alpha\beta)\,b_0 + \alpha m_0 a_0 = 0, \end{aligned} \tag{14}$$

and

$$\begin{aligned} &(n+3)\alpha^2\beta^2 a_{n+3} - \alpha\beta\,[2n+5+k+\alpha\beta]\,a_{n+2} \\ &\quad + [n+2+k+2\alpha\beta]\,a_{n+1} - a_n + \alpha m_0 b_{n+1} \\ &\quad - \frac{(E+m_0)}{\beta}b_n = 0, \\ &(n+3)\alpha^2\beta^2 b_{n+3} - \alpha\beta\,[2n+5-k+\alpha\beta]\,b_{n+2} \\ &\quad + [n+2-k+2\alpha\beta]\,b_{n+1} - b_n + \alpha m_0 a_{n+1} \\ &\quad + \frac{(E-m_0)}{\beta}a_n = 0, \end{aligned} \tag{15}$$

with $a_0, a_1, a_2, b_0, b_1, b_2 \neq 0$. From these relations, one obtains $\beta = \sqrt{m^2 - E^2}$, which determines the factor $\mathrm{e}^{-\sqrt{m^2-E_N^2}\,r}$ in the wave functions (12) – this leads to the same behavior as that obtained with the Dirac equation. The relation

$$[N + 2\alpha\beta]\,\beta - \alpha m^2 = 0 \tag{16}$$

is also obtained from (13)–(15) and gives the relation for the energy levels

$$E_N = \pm m_e c^2 \sqrt{\frac{1}{2} - \frac{N^2}{8\alpha^2} \pm \frac{N}{4\alpha}\sqrt{\frac{N^2}{4\alpha^2} + 2}}, \quad (17)$$

where the physical values are the positive ones. This spectrum may be compared with the one obtained with the Dirac equation [22,23] (also deduced by Sommerfeld [25])

$$E_N = \frac{m_e c^2}{\sqrt{1 + \alpha^2/a^2}}, \quad (18)$$

where $a = N - j + 1/2 + \sqrt{(j+1/2)^2 - \alpha^2}$, if one takes numerical results. For example, for the electron–proton interactions in the deuterium atom, the results may be obtained considering the fine-structure constant [26] $\alpha = 1/137.03599976$, $Z = 1$ and the electron mass $m_0 = 0.510998902\,\mathrm{MeV}/c^2$ [26]. The experimental ground state energy for the deuterium atom is $E_1 = -13.60214\,\mathrm{eV}$[1]; calculating it with the Dirac spectrum (18), one has $-13.60587\,\mathrm{eV}$, and with (17), $-13.60298\,\mathrm{eV}$. More numerical results may be found in [18]. Observing these results, one can see that the accord of both theories with the deuterium experimental data is very good, but the results from (17) are closer to the experimental data than the results from (18). The results form the Dirac theory (18); one can see that the deviations from the data are of the order of 0.027%. Considering the spectrum of (17), the deviations are of the order of 0.005%, almost five times smaller, which is a significant improvement. The same pattern occurs when the other energy levels are compared [18].

3 Quark confinement

At this point, it is possible to use the theory proposed in [18] and briefly exposed in the latter section in order to study the hadronic structure. If one considers the hadronic structure in a way similar to the Bogolioubov [13] and MIT bag [14] models, supposing that the hadrons are composed of constituent quarks, which generate a self-consistent color field, and that this field may be described by a spherical potential, (9) may be used. In this section, some results and the implications of these ideas in studying quark confinement will be discussed.

In a baryon, for example, that is a qqq system, the wave functions and energies of the individual constituent quarks are determined by (9). So, the mass of the hadron is the sum of the individual energies $E_{i,n}$ of the i quark at state n:

$$M = \sum_{i=1}^{3} E_{i,n}. \quad (19)$$

In a nucleon, the three quarks are supposed to be in the ground state E_1. If we also suppose that $m_u = m_d = m$, the nucleon mass will be $M_N = 3E_1$. In this framework, excited states of the quarks produce hadrons in higher energy states and resonances.

Now, as an example, let us consider, for simplicity, that the field generated by the quarks has a shape similar to a strong Coulomb potential (for strong interactions, a strong Coulomb potential, with $\alpha \sim 1$, may be considered). In this case, inside the classical horizon of events, for $r < r_0$, the quark wave functions are given by (12) with energies (17). But near the horizon of events, at $r = r_0$ these solutions are not valid, and for this reason, this case must be studied separately. The solution of the equation in the neighborhood of r_0, may be given by an expression similar to (12), but replacing ρ for $\rho - \alpha\beta/m_0$,

$$F = \rho^s \sum_{n=0}^{N} a_n \left(\rho - \frac{\alpha\beta}{m_0}\right)^n \mathrm{e}^{-\rho},$$
$$G = \rho^s \sum_{n=0}^{N} b_n \left(\rho - \frac{\alpha\beta}{m_0}\right)^n \mathrm{e}^{-\rho}. \quad (20)$$

With this procedure, one finds that near the horizon of events, just one energy value is possible, $E = 0$. The other conditions for the existence of a solution are $k = 0$ ($l = 0$) and $s = -1 - \alpha$, which means an infinite discontinuity of the wave function (20) at $r = r_0$. If this solution is discarded, the trivial solution $\psi(r_0) = 0$ must be considered, which can be interpreted as a boundary condition at $r = r_0$. Consequently, this solution tells us that the space is divided in two parts (a fact that is true in both cases) inside and outside the horizon, which does not communicate. So, at $r \sim r_0$, the solution (20), imposes the confinement of quarks inside this region. Classically thinking, the quarks are confined by a trapping surface [28] that is generated by the potential.

Considering the Υ meson family, that is composed by $b\bar{b}$ states, the theory may be applied just considering b constituent quarks with masses $m_b = 5.5\,\mathrm{GeV}$, and a Coulomb potential with $\alpha = 1.05$, which is a reasonable value for quark interactions. From the expression (17) one has

$$M_{i,j} = m_b \sum_{n=i,j} \sqrt{\frac{1}{2} - \frac{n^2}{8\alpha^2} + \frac{n}{4\alpha}\sqrt{\frac{n^2}{4\alpha^2} + 2}}, \quad (21)$$

and consequently, $m_\Upsilon = M_{1,1} = 9.47\,\mathrm{GeV}$ (the experimental value is $m_\Upsilon = 9.46\,\mathrm{GeV}$ [26]). Excited states of the quarks determine the other Υ states, up to the limit value $m_{\max} = 2m_b = 11.0\,\mathrm{GeV}$; this is the mass of the last Υ state found in [26], the $\Upsilon(11020)$. The first excited state is $M_{1,2} = 9.83\,\mathrm{GeV}$, which shows a small discrepancy of the experimental value (less than 2%), that is 10.02 GeV. The second state is $M_{2,2} = 10.19\,\mathrm{GeV}$, which is still close to the experimental value, 10.35 GeV [26]. The quarks will be confined inside the region $r < r_0 = 0.05\,\mathrm{fm}$, which is a reasonable size for the core of a meson. Some estimates of these quantities for other hadrons may be found in Table 1 [18].

One must remark that in order to describe the spectra of the particles of Table 1 more accurately, specially in the

[1] The experimental values for the energy levels of the hydrogen and deuterium may be found in [24].

Table 1. Values of the masses M of the hadrons, composed of constituent quarks of mass m compared with the experimental ones [26]. The calculations are made with (17), obtained for Coulomb potentials with coupling α

	m (GeV)	α	r_0 (fm)	M (GeV)	$M_{\rm exp}$ (GeV)
N(qqq)	0.38	1.60	0.83	0.938	0.938 (proton)
$J/\psi(c\bar{c})$	1.79	1.00	0.11	3.10	3.10
$\Upsilon(b\bar{b})$	5.50	1.05	0.05	9.47	9.46

excited states, other terms must be added in the potential, or even a better shape for the potential, based on the QCD, must be considered. This fact (that is widely used [6–12]) may be understood if one observes that for short-range interactions many effects may occur, generating corrections to the potential. Another factor that must be considered to improve the description is that spherical symmetry is not the best one for $q\bar{q}$ mesons, and this fact must be corrected in future works.

In this description, the horizon of events is not an exclusive feature of the Coulomb potential; it may appear for any attractive potential, when the condition

$$\xi(r_0) = 0, \tag{22}$$

is satisfied, which occurs for $V(r_0) = -mc^2$. In these cases, the only energy value is $E = 0$, and the solution will present the discontinuity shown above,

$$\psi \propto \frac{f(r)}{(r - r_0)^{\delta}} \tag{23}$$

with $\delta > 0$, results that lead to quark confinement for general attractive potentials. So, as we can see, with this description, quark confinement is a phenomenon that happens in a natural way for many kinds of potentials.

An interesting question that may be asked is how a particle, as for example an electron, can probe the internal structure of the hadron entering and leaving the black hole. To answer this apparently paradoxical question, one must observe that this black hole is generated by the strong (color) interactions, and the leptons are not affected by strong forces. So, the black hole does not exist for this kind of particles, and the leptons may probe the internal structure of the hadrons if their energy is high enough. This is a good reason to use leptons to investigate the interior of the hadrons. The radiative decays of hadrons $h \to h'\gamma$ may be understood in a similar way. Inside the hadron h an excited quark q^* may decay by the process $q^* \to q\gamma$ and the produced γ may be observed, as it is not affected by the black hole.

Hadronic decays of the type $h \to h' +$ hadrons may occur when the excited quarks are in states where the wave functions are not small near r_0. In these cases, when the quarks are near the black hole surface at $r \sim r_0$, they may excite the vacuum, creating a $q\bar{q}$ pair and decaying in a state where $\psi(r \sim r_0)$ is smaller. A similar situation is expected classically; Hawking in [27] proposed that particles may be created thermally near black holes. The probability of creating particles is related to the part of the wave function that would leave the black hole [27], so, smaller values of $\psi(r \sim r_0)$ means a smaller probability of creating a $q\bar{q}$ pair, and consequently of decay of the hadron. So, as we can see, in the quantum world, black holes are not so black, and there are many processes where the particles may reach the external world.

4 Discussion of the results

In this work it was shown how quark confinement appears when relativistic wave equations in curved spaces are used. Now the obtained results will be compared with the results of the existing models.

As it was said in the introduction, many authors succeeded in explaining quark confinement with phenomenological potentials. In the non-relativistic oscillator model [5–7], the Hamiltonian (1) leads to confining oscillator-type wave functions that contain a factor $e^{-\beta(r_1^2+r_2^2+r_3^2)}$, where β is a constant. In [17] with the potential (4), the approximate behavior of the wave function is $\psi \propto \Phi(r)e^{-\beta r^2}$ and in the Cornell model [9], where the potential of the type (2) is used, a similar behavior occurs. In the Bogolioubov and in the MIT bag models that consider a potential of the type (3), the wave functions contain a factor $e^{-\beta\sqrt{m^2-E^2}(r-r_0)}$, and quark confinement appears in the limit $V_0 \to \infty$, where the wave function is constant for $r = R$ and 0 for $r > R$.

As it was seen in the previous section, quark confinement appears in a different way when the equations derived in curved spaces are used. Differently from the other models, for an internal particle it is not possible to reach the surface $r = r_0$, as $\psi(r_0) = 0$. One must observe that even in the MIT bag model, with an infinite potential, so strong a condition is not reached. The result of this condition is that the space-time is divided in two disconnected regions, inside and outside the surface. This fact is an intrinsic property of the space-time, due to attractive potentials, as for example the Coulomb potential, and there is no need of introducing confining potentials to obtain this effect. In fact, potentials of the type (1)–(4) represent in an approximate way, in plane space-time formulations, systems that are described in a natural way by curved spaces.

Another interesting aspect of the theory is that classically it is expected that a collapse occurs at the origin, but here we are dealing with quantum systems, and the uncertainty principle forbids this collapse. The solution of the equation shows that the wave function is 0 at the origin, confirming this statement.

The Dirac theory [29,30] introduced the special relativity in quantum mechanics, so it is very reasonable to think that the next step is to formulate the quantum mechanics in terms of the general relativity ideas. The deuterium spectrum obtained in this way shows that the corrections of the energy levels, due to this general formulation of quantum mechanics (or general quantum mechanics) with the inclusion of the electric curvature of the space-time, provide a quite impressive agreement with the experimental data. The same agreement is expected to happen in hadron spectroscopy with the introduction of the correct shape of the potential. Another piece of strong evidence in the validation of the theory is the quark confinement mechanism proposed in this paper, where the quarks are confined by a trapping surface, similar to the one defined by Penrose [28]. Conceptually, these are very important results, as they show a successful way to join quantum mechanics and general relativity.

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